Exercise 52

If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V(t) = 100{,}000\left(1 - \frac{1}{60}t\right)^2 \qquad 0 \le t \le 60$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) as a function of t. What are its units? For times $t = 0$, 10, 20, 30, 40, 50, and 60 min, find the flow rate and the amount of water remaining in the tank. Summarize your findings in a sentence or two. At what time is the flow rate the greatest? The least?

Solution

The rate at which the water is flowing out of the tank is the derivative of V.

$$\begin{aligned}
V'(t) &= \lim_{h\to 0} \frac{V(t+h) - V(t)}{h} \\
&= \lim_{h\to 0} \frac{100{,}000\left[1 - \frac{1}{60}(t+h)\right]^2 - 100{,}000\left(1 - \frac{1}{60}t\right)^2}{h} \\
&= 100{,}000 \lim_{h\to 0} \frac{\left[\left(1 - \frac{1}{60}t\right) - \frac{1}{60}h\right]^2 - \left(1 - \frac{1}{60}t\right)^2}{h} \\
&= 100{,}000 \lim_{h\to 0} \frac{\left[\left(1 - \frac{1}{60}t\right)^2 - \frac{1}{30}h\left(1 - \frac{1}{60}t\right) + \frac{1}{60^2}h^2\right] - \left(1 - \frac{1}{60}t\right)^2}{h} \\
&= 100{,}000 \lim_{h\to 0} \frac{-\frac{1}{30}h\left(1 - \frac{1}{60}t\right) + \frac{1}{60^2}h^2}{h} \\
&= 100{,}000 \lim_{h\to 0} \left[-\frac{1}{30}\left(1 - \frac{1}{60}t\right) + \frac{1}{60^2}h\right] \\
&= 100{,}000 \left[-\frac{1}{30}\left(1 - \frac{1}{60}t\right)\right] \\
&= -\frac{10{,}000}{3} + \frac{500}{9}t
\end{aligned}$$

The units of $V'(t)$ are gallons per minute.

t	$V(t)$	$V'(t)$
0	100 000	−3333
10	69 444	−2778
20	44 444	−2222
30	25 000	−1667
40	11 111	−1111
50	2778	−556
60	0	0

The more water there is in the tank, the higher the flow rate is out of the tank.